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Chebychev Subspaces in the Space of Bounded Linear Operators from c_0 to c_0

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1. INTRODUCTION

The space c_0 is the linear space of all sequences $x = \{x_j\}$ converging to zero, with the norm of x given by $\| x \| = \sup | x_j |$, the supremum taken over all j. The symbol $[c_0, c_0]$ will denote the linear space of all bounded linear operators from c_0 to c_0. If $T \in [c_0, c_0]$, then the norm of T is the standard operator norm given by

$$\| T \| = \sup\{\| T(x)\| : x \in c_0, \| x \| \leqslant 1\}.$$

If M is a subset of the normed linear space X and $x \in X$, then a point x_0 in M is said to be a *best approximation* to x from M if $\| x - x_0 \| = \inf\{\| x - y \| : y \in M\}$. If each x in X has a unique best approximation in M, then M is called a *Chebychev* subset of X.

In this paper we are concerned with the characterization of best approximations in a finite dimensional subspace M of $[c_0, c_0]$, and the determination of conditions under which M is Chebychev. An element x in X has x_0 as a best approximation in a subspace M if and only if $x - x_0$ has 0 as a best approximation in M. Therefore, to characterize best approximations in M, it suffices to provide conditions under which an element has 0 as a best approximation in M. It is known (see, e.g., [2, p. 20]) that if M is a finite dimensional subspace of X, then each $x \in X$ has a best approximation in M. Thus, if M is non-Chebychev, there exists some element $x \in X$ with two best approximations in M.

By a result in [5], each bounded linear operator in $[c_0, c_0]$ can be represented by an infinite matrix of scalars. We use this fact in Section 2 to characterize best approximations in a finite dimensional subspace of $[c_0, c_0]$. In

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Section 3 we will characterize one-dimensional Chebychev subspaces in $[c_0, c_0]$, and in Section 4 present a necessary condition and a sufficient condition for a finite dimensional subspace of $[c_0, c_0]$ to be non-Chebychev. Finally, we will show that if a bounded linear operator T in $[c_0, c_0]$ is represented by an infinite matrix, then the second adjoint T^{**} in $[l_\infty, l_\infty]$ may also be represented by that matrix. This permits the reformulation of previous results in terms independent of the operator's matrix representation.

Unless otherwise stated, all notation will correspond to that of [3]. All scalars will be assumed to be real. Let X be a linear space with norm $\|\cdot\|$. The conjugate space X^* will be assumed to have the usual operator norm. For each x in X, $\hat{x}$ will denote that functional in X^{**} defined by $\hat{x}(x^*) = x^*(x)$ for all x^* in X^*, and $\hat{X} = \{\hat{x}: x \in X\}$. The norm closed unit sphere of X will be denoted by $S(X)$. For any set A, $\mathrm{cl}(A)$ will mean the norm closure of A. The *annihilator* $M^\perp$ of a subspace M of X is defined by

$$M^\perp = \{x^* \in X^*: x^*(y) = 0 \text{ for all } y \in M\}.$$

If $x_1, \ldots, x_n$ are vectors in the linear space X, then $[x_1, \ldots, x_n]$ will denote the linear subspace of X spanned by these vectors. We will assume, unless otherwise stated, that $[x_1, \ldots, x_n]$ has dimension n. If Y is a normed linear space, then by $(Y \times \cdots \times Y)_\infty$ (n summands), we will mean the linear space of all ordered n-tuples of the form $y = (y_1, \ldots, y_n)$ for $y_i \in Y$, $i = 1, \ldots, n$ with norm defined by $\|y\| = \max\{\|y_i\|: 1 \leqslant i \leqslant n\}$. The symbol $(Y \times \cdots \times Y)_1$ (n summands) is defined similarly, with the norm in this case defined by $\|y\| = \sum_{i=1}^{n} \|y_i\|$. The following lemma is then easily seen.

LEMMA 1.1. *Let Y be a normed linear space. If for $f = (f_1, \ldots, f_n) \in (Y^* \times \cdots \times Y^*)_1$ (n summands), we write $f(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$, for all $(x_1, \ldots, x_n) \in E = (Y \times \cdots \times Y)_\infty$ (n summands), then E^* can be identified with $(Y^* \times \cdots \times Y^*)_1$ (n summands).*

We state here for convenience a known result which may be found in [4].

THEOREM 1.2. *Let M be a subspace of the normed linear space X, and let $x \in X - \mathrm{cl}(M)$. Then x has 0 as a best approximation in M if and only if there exists f in $M^\perp$, $\|f\| = 1$, such that $f(x) = \|x\|$.*

2. CHARACTERIZATION OF BEST APPROXIMATIONS

Each bounded linear operator in $[c_0, c_0]$ may be represented by an infinite matrix of scalars according to the following theorem found in [5, p. 217].

THEOREM 2.1. *If $A \in [c_0, c_0]$, there exists a unique infinite matrix of scalars (a_{ij}) $(i, j = 1, 2, \ldots)$ such that*

(i) $\| A \| = \sup_i \sum_{j=1}^{\infty} | a_{ij} |$,

(ii) $\lim_{i\to\infty} a_{ij} = 0$, $j = 1, 2, \ldots$,

(iii) *if* $x = \{x_i\}$, $y = \{y_i\} \in c_0$ *with* $y = Ax$, *then* $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$, $i = 1, 2, \ldots$.

Conversely, if (a_{ij}) $(i, j = 1, 2, \ldots)$ is an infinite matrix of scalars such that $\sup \sum_{j=1}^{\infty} | a_{ij} |$ *(supremum taken over i) is finite and such that* (ii) *holds, then the equation in* (iii) *defines a member A of $[c_0, c_0]$ whose norm is given by* (i).

A problem in considering the space $[c_0, c_0]$ is that there is no convenient way to represent the bounded linear functionals on the space. However, if we restrict the matrices in $[c_0, c_0]$ to a fixed finite number of rows, we obtain the restricted space $E = (l_1 \times \cdots \times l_1)_\infty$. It is known (see [3]) that $l_1{}^*$ can be identified with l_∞. Then since E^* can be identified with $(l_\infty \times \cdots \times l_\infty)_1$ by Lemma 1.1, we know what form the bounded linear functionals on the restricted space take. Hence, to obtain some of the following results, we consider a selected finite number of rows in the matrices.

In order to characterize best approximations in a finite dimensional subspace M of $[c_0, c_0]$, we will need the next lemma. If K is a set of positive integers and $A = (a_{ij})$ is an infinite matrix, then $A \mid K$ denotes the matrix whose rows are precisely the rows (a_{ij}) $(j = 1, 2, \ldots)$ of A for which $i \in K$.

LEMMA 2.2. *Let $A_1, \ldots, A_n$ be linearly independent operators in $[c_0, c_0]$ with $M = [A_1, \ldots, A_n]$. Then*

(a) *there exists a positive integer p such that if $K_p = \{i: 1 \leqslant i \leqslant p\}$ and $\bar{A}_i = A_i \mid K_p \in (l_1 \times \cdots \times l_1)_\infty$ (p summands), $i = 1, \ldots, n$, then $\bar{A}_1, \ldots, \bar{A}_n$ are linearly independent.*

(b) *given $B \in [c_0, c_0]$, there exists a nonnegative constant Q such that for any positive integer $s \geqslant p$, if $i_1 < \cdots < i_{s-p}$ denote any fixed positive integers with $i_1 > p$, $K_s = \{i: 1 \leqslant i \leqslant p$ or $i = i_1, \ldots, i_{s-p}\}$, $\bar{A}_i{}^s = A_i \mid K_s$, $\bar{B}^s = B \mid K_s$, and $\bar{A}^s = \sum_{i=1}^{n} \lambda_i \bar{A}_i{}^s$ is a best approximation to $\bar{B}^s$ in $[\bar{A}_1{}^s, \ldots, \bar{A}_n{}^s]$, then we have $| \lambda_i | \leqslant Q$, $i = 1, \ldots, n$.*

Proof. For each positive integer n, let $K_n = \{i: 1 \leqslant i \leqslant n\}$ and define the mapping φ_n on M by

$$\varphi_n(A) = \bar{A}^n = A \mid K_n \qquad \text{for} \qquad A \text{ in } M.$$

We have $\bar{A}^n \in (l_1 \times \cdots \times l_1)_\infty$ (n summands). For any n, φ_n is a bounded linear transformation and is, thus, continuous. Let $A = (a_{ij}) \in M$ with

$\| A \| = 1$. Then there exists $i = n_A$ such that $\sum_{j=1}^{\infty} | a_{n_A j} | > (1/2)$. Therefore, $\| \varphi_{n_A}(A) \| > (1/2)$. Let $\mathcal{O} = \{\varphi_{n_A}(C): C \in M \text{ with } \| \varphi_{n_A}(C) \| > (1/2)\}$. Let $U_A = \varphi_{n_A}^{-1}(\mathcal{O})$, so U_A is open in M. Let $M' = \{A \text{ in } M: \| A \| = 1\}$. Then M' is a closed subset of $S(M)$, which is compact since M is finite dimensional. Therefore, M' is compact. Since $\{U_A : A \in M'\}$ is an open covering of M', there exists a finite subcover $\{U_{B_1}, \ldots, U_{B_\ell}\}$ of M' for $B_1, \ldots, B_\ell$ in M'. Let $p = \max\{n_{B_1}, \ldots, n_{B_\ell}\}$. If $A \in M'$, then $A \in U_{B_i}$ for some $i = 1, \ldots, \ell$. Therefore, $\| \varphi_p(A) \| > (1/2)$. Hence, for all A in M', we have $\| \varphi_p(A) \| > (1/2)$.

Now $\varphi_p(A_i) = \bar{A}_i^p = \bar{A}_i$, $i = 1, \ldots, n$. Suppose $\bar{A}_1, \ldots, \bar{A}_n$ are linearly dependent. Then there exists A in M, $A \neq 0$, such that $\varphi_p(A) = 0$. However, $A/\| A \| \in M'$, so $\| \varphi_p(A/\| A \|) \| > (1/2)$, a contradiction. Therefore, $\bar{A}_1, \ldots, \bar{A}_n$ are linearly independent, and (a) is proved.

To prove (b), let φ_p be defined on M by $\varphi_p(A) = \bar{A} = A \mid K_p$ for A in M. Then φ_p is a continuous linear transformation from M onto $[\bar{A}_1, \ldots, \bar{A}_n]$. Also φ_p is one-to-one since $\bar{A}_1, \ldots, \bar{A}_n$ are linearly independent by (a). Thus, φ_p has an inverse φ_p^{-1} which is clearly a linear transformation. By the open mapping theorem, φ_p^{-1} is continuous and, hence, bounded.

Now define a new norm $\| \cdot \|_1$ on $[A_1, \ldots, A_n]$ by

$$\left\| \sum_{i=1}^{n} \delta_i A_i \right\|_1 = \max | \delta_i |,$$

where the maximum is taken over $1 \leqslant i \leqslant n$. Since in a finite dimensional space all norms are equivalent, there exists a positive constant c such that $\| A \|_1 \leqslant c \| A \|$ for all A in M. Let $B \in [c_0, c_0]$. Then let $Q = 2c \| \varphi_p^{-1} \| \| B \|$.

Let s be a positive integer such that $s \geqslant p$, and let $\bar{A}_i^s$, $\bar{B}^s$, and $\bar{A}^s$ be as given in (b). Define φ_s on M by $\varphi_s(C) = \bar{C}^s = C \mid K_s$ for any C in M. We can easily see that $\| \varphi_s^{-1} \|$ exists and $\| \varphi_s^{-1} \| \leqslant \| \varphi_p^{-1} \|$. Since $\bar{A}^s$ is a best approximation to $\bar{B}^s$ in $[\bar{A}_1^s, \ldots, \bar{A}_n^s]$, we have $\| \bar{A}^s \| \leqslant \| \bar{B}^s - \bar{A}^s \| + \| \bar{B}^s \| \leqslant 2 \| B \|$. Thus,

$$\max | \lambda_i | = \left\| \sum_{i=1}^{n} \lambda_i A_i \right\|_1 = \| A \|_1 \leqslant c \| A \|$$

$$\leqslant c \| \varphi_s^{-1} \| \| \bar{A}^s \| \leqslant 2c \| \varphi_p^{-1} \| \| B \| = Q.$$

Therefore, $| \lambda_i | \leqslant Q$, $i = 1, \ldots, n$, and (b) is proved.

Now we are ready to give necessary and sufficient conditions for an element to have 0 as a best approximation in a finite dimensional subspace of $[c_0, c_0]$. Without loss of generality, we may assume that each of the operators generating the subspace has norm equal to 1.

THEOREM 2.3. *Let* $A_k = (a_{ij}^k) \in [c_0, c_0]$ *with* $\| A_k \| = 1$, $k = 1, \ldots, n$, *and* let $B = (b_{ij}) \in [c_0, c_0]$. *Then* B *has* 0 *as a best approximation in* $[A_1, ..., A_n]$ *if and only if for all* $\epsilon > 0$, *there exist* m *positive integers* $k_1, ..., k_m$, m l_∞ *sequences* $f^1, ..., f^m$ *with* $\| f^i \| = 1$, $i = 1, ..., m$ *and* m *scalars* $r_1, ..., r_m$ *with* $r_i > 0$, $i = 1, ..., m$ *and* $\sum_{i=1}^m r_i = 1$ *such that*

(i) $\sum_{i=1}^m r_i \sum_{j=1}^\infty f_j^i a_{k_i j}^k = 0 \qquad k = 1, ..., n,$

(ii) $| \sum_{i=1}^m r_i \sum_{j=1}^\infty f_j^i b_{k_i j} - \| B \| | < \epsilon.$

Proof. Choose p and Q as in Lemma 2.2. Suppose B has 0 as a best approximation in $[A_1, ..., A_n]$. Let $\lambda_1, ..., \lambda_n \in [-Q, Q]$ and $\epsilon > 0$. Then there exists $i = k(\lambda_1, ..., \lambda_n) = k(\lambda)$ such that

$$\Big| \sum_{j=1}^{\infty} | b_{k(\lambda)j} - (\lambda_1 a_{k(\lambda)j}^1 + \cdots + \lambda_n a_{k(\lambda)j}^n)| - \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| \Big| < (\epsilon/6). \tag{1}$$

Let $\mu_1, ..., \mu_n \in [-Q, Q]$. It is easily seen that the function $\varphi(\mu_1, ..., \mu_n) = \sum_{j=1}^\infty | b_{k(\lambda)j} - (\mu_1 a_{k(\lambda)j}^1 + \cdots + \mu_n a_{k(\lambda)j}^n)|$ is continuous at $(\lambda_1, ..., \lambda_n)$. Therefore, for each $i = 1, ..., n$ there exists an open interval

$$I_{\lambda_i} = \{\mu : | \mu - \lambda_i | < (\epsilon/6n)\}$$

such that for $\mu_1, ..., \mu_n \in [-Q, Q]$, if $\mu_i \in I_{\lambda_i}$ for each $i = 1, ..., n$, then

$$\Big| \sum_j | b_{k(\lambda)j} - (\mu_1 a_{k(\lambda)j}^1 + \cdots + \mu_n a_{k(\lambda)j}^n)| - \sum_j | b_{k(\lambda)j} - (\lambda_1 a_{k(\lambda)j}^1 + \cdots + \lambda_n a_{k(\lambda)j}^n)| \Big| < (\epsilon/6). \tag{2}$$

Now using (1) and (2) we obtain for $\mu_1, ..., \mu_n \in [-Q, Q]$ and $\mu_i \in I_{\lambda_i}$, $i = 1, ..., n$

$$\Big| \sum_j | b_{k(\lambda)j} - (\mu_1 a_{k(\lambda)j}^1 + \cdots + \mu_n a_{k(\lambda)j}^n)| - \| B - (\mu_1 A_1 + \cdots + \mu_n A_n)\| \Big| < (\epsilon/2). \tag{3}$$

For a scalar $\lambda \in [-Q, Q]$, let $I_\lambda = \{\mu : | \mu - \lambda | < (\epsilon/6n)\}$. Then $\{I_\lambda : \lambda \in [-Q, Q]\}$ is an open covering of the compact interval $[-Q, Q]$. Consequently, there exists a finite number of scalars $\lambda_1, ..., \lambda_s$ in $[-Q, Q]$ such that $[-Q, Q] \subseteq \bigcup_{\ell=1}^s I_{\lambda_\ell}$. Consider $k(\lambda_p) = k(\lambda_{p_1}, ..., \lambda_{p_n})$, where λ_{p_i} may be chosen from λ_1 to λ_s for $i = 1, ..., n$. Thus, we have s^n rows. Now add to these the first p rows as obtained by Lemma 2.2 (a). Let m be the number of distinct rows obtained, so $m \leqslant p + s^n$. Label these rows by $k_1, ..., k_m$.

Let $\lambda_1, ..., \lambda_n$ be arbitrary scalars in $[-Q, Q]$. Then for each $i = 1, ..., n$, $\lambda_i \in I_{\lambda_{\ell_i}}$ for some $\ell_i = 1, ..., s$. Now $k(\lambda_{\ell_1}, ..., \lambda_{\ell_n}) = k_\ell$ for some $\ell = 1, ..., m$. Then we may apply (3) to obtain

$$\left| \sum_j | b_{k_\ell j} - (\lambda_1 a^1_{k_\ell j} + \cdots + \lambda_n a^n_{k_\ell j})| - \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| \right| < (\epsilon/2).$$

It follows that

$$\left| \max_{1 \leqslant \ell \leqslant m} \sum_j | b_{k_\ell j} - (\lambda_1 a^1_{k_\ell j} + \cdots + \lambda_n a^n_{k_\ell j})| \right.$$
$$\left. - \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| \right| < (\epsilon/2). \tag{4}$$

Define, respectively, $\bar{A}_1, ..., \bar{A}_n$, $\bar{B} = A_1, ..., A_n, B \mid$ rows $k_1, ..., k_m \in E = (l_1 \times \cdots \times l_1)_\infty$ (m summands). By Lemma 2.2 $\bar{A}_1, ..., \bar{A}_n$ are linearly independent. By (4) we obtain

$$|\| \bar{B} - (\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n)\| - \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\|| < (\epsilon/2). \tag{5}$$

Consider the quotient space $E/[\bar{A}_1, ..., \bar{A}_n]$ with the quotient mapping π. We have $\| \pi \bar{B} \| = \inf \| \bar{B} - (\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n)\|$, where the infimum is taken over $\lambda_i \in [-Q, Q]$, $i = 1, ..., n$ by Lemma 2.2 (b). We know $\| \pi \bar{B} \| \leqslant \| \bar{B} \| \leqslant \| B \|$, so $\| \pi \bar{B} \| - \| B \| < \epsilon$. Utilizing (5) and the fact that B has 0 as a best approximation in $[A_1, ..., A_n]$, we see that

$$\| B \| \leqslant (\epsilon/2) + \inf \| \bar{B} - (\lambda_1 \bar{A}_1 + \cdots + \lambda_n \bar{A}_n)\|,$$

the infimum taken over $\lambda_i \in [-Q, Q]$, $i = 1, ..., n$. Thus, $\| B \| - \| \pi \bar{B} \| < \epsilon$. Hence,

$$|\| \pi \bar{B} \| - \| B \|| < \epsilon. \tag{6}$$

Suppose $B \neq 0$. Then by the Hahn–Banach theorem there exists F in E^*, $\| F \| = 1$, such that $F([\bar{A}_1, ..., \bar{A}_n]) = 0$ and $F(\bar{B}) = \| \pi \bar{B} \|$. By Lemma 1.1, E^* can be identified with $(l_\infty \times \cdots \times l_\infty)_1$ (m summands). Hence, F can be represented by an m-tuple of l_∞ sequences $(g^1, ..., g^m)$ where $1 = \| F \| = \sum_{i=1}^m \| g^i \|$. Without loss of generality, assume $\| g^i \| > 0$ for $i = 1, ..., m$. Then define

$$f^i = g^i / \| g^i \|, \qquad r_i = \| g^i \| \qquad i = 1, ..., m.$$

Then $\sum_{i=1}^m r_i = 1$ and $\| f^i \| = 1$ for $i = 1, ..., m$. For $T = (t_{ij}) \in [c_0, c_0]$, $F(\bar{T}) = \sum_{i=1}^m r_i \sum_{j=1}^\infty f_j^i t_{k_i j}$. Then (i) holds since $F([\bar{A}_1, ..., \bar{A}_n]) = 0$. Since $F(\bar{B}) = \| \pi \bar{B} \|$, we have $| F(\bar{B}) - \| B \|| < \epsilon$ by (6). Hence, (ii) holds.

Now if $B = 0$, select $\bar{C} \in E, \bar{C} \notin [\bar{A}_1, ..., \bar{A}_n]$. Again apply the Hahn–Banach theorem to obtain F in E^*, $\| F \| = 1$ such that $F([\bar{A}_1, ..., \bar{A}_n]) = 0$ and $F(\bar{C}) = \| \pi \bar{C} \|$. By the same argument as above, (i) holds, and (ii) holds since $\bar{B} = 0 = \| B \|$.

To show sufficiency, let $\epsilon > 0$. Then there exist k_i, f^i, and r_i, $i = 1, ..., m$ as stated in the theorem such that (i) and (ii) hold. For $T = (t_{ij}) \in [c_0, c_0]$, define F on $[c_0, c_0]$ by

$$F(T) = \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j^i t_{k_i j} .$$

Now $| F(T)| \leqslant \| T \|$, so $F \in [c_0, c_0]^*$ and $\| F \| \leqslant 1$.

We have $F(A_k) = 0$ for $k = 1, ..., n$ by (i), and $| F(B) - \| B \|| < \epsilon$ by (ii). Let $\lambda_1, ..., \lambda_n$ be arbitrary scalars. Then $F[B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)] = F(B)$. Hence,

$$\| B \| - \epsilon < F[B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)] \leqslant \| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\|.$$

But this can be shown for all $\epsilon > 0$. Therefore,

$$\| B - (\lambda_1 A_1 + \cdots + \lambda_n A_n)\| \geqslant \| B \|.$$

Since the scalars were arbitrary, B has 0 as a best approximation in $[A_1, ..., A_n]$. This completes the proof.

3. Characterization of One-Dimensional Chebychev Subspaces

Definition. If P_1 and P_2 are subsets of the set I of all positive integers such that $P_1 \cap P_2 = \phi$ and $P_1 \cup P_2 = I$, then we say that P_1 and P_2 form a *partition* of I. Let $A = (a_{ij}) \in [c_0, c_0]$. Then A satisfies the *partition property* if and only if there exists $\delta > 0$ such that for all i and for all partitions P_1, P_2 of I, we have

$$\left| \sum_{j \in P_1} | a_{ij} | - \sum_{j \in P_2} | a_{ij} | \right| \geqslant \delta.$$

We will show that if $A \neq 0$, then $[A]$ is a Chebychev subspace of $[c_0, c_0]$ if and only if A satisfies the partition property. Before we can prove this result, however, we will need two lemmas.

Lemma 3.1. *Let $A = (a_{ij}) \in [c_0, c_0]$. Then the partition property fails to hold for A if and only if either there exists i and a partition P_1, P_2 of I such that*

$$\left| \sum_{j \in P_1} | a_{ij} | - \sum_{j \in P_2} | a_{ij} | \right| = 0,$$

or there exists a sequence $i_1 < \cdots < i_n < \cdots$ *and corresponding partitions* $P_1{}^n$, $P_2{}^n$ *such that* $| \sum_{j \in P_1{}^n} | a_{i_n j} | - \sum_{j \in P_2{}^n} | a_{i_n j} || < (1/n)$.

Proof. If either of the conditions hold, A clearly fails to satisfy the partition property.

Conversely, suppose the partition property fails to hold for A. To simplify notation in this proof, let $Q(i, P_1, P_2) = | \sum_{j \in P_1} | a_{ij} | - \sum_{j \in P_2} | a_{ij} ||$. If the first condition holds, we are finished. Therefore, assume there does not exist i and a partition P_1, P_2 such that $Q(i, P_1, P_2) = 0$. Since the partition property fails to hold for A, there exists i and P_1, P_2 such that $Q(i, P_1, P_2) < 1$. Let i_1 be the smallest such i where this occurs, i.e., for all $i < i_1$ and for all P_1, P_2 we have $Q(i, P_1, P_2) \geqslant 1$, but there exists a partition $P_1{}^1$, $P_2{}^1$ such that $Q(i_1, P_1{}^1, P_2{}^1) < 1$.

Now there exists i and P_1, P_2 such that $Q(i, P_1, P_2) < (1/2)$. We know $i < i_1$ is impossible. Suppose the only possible choice is $i = i_1$. Then we must have one row i and a sequence of partitions $P_1{}^n$, $P_2{}^n$ such that $Q(i, P_1{}^n, P_2{}^n) < (1/n)$. But we will show that this is impossible.

For each n, define $g^n = \{g_j{}^n\}$ by

$$g_j{}^n = \begin{cases} 1 & \text{for } j \in P_1{}^n, \quad a_{ij} \geqslant 0, \quad \text{or} \quad j \in P_2{}^n, \quad a_{ij} < 0, \\ -1 & \text{for } j \in P_1{}^n, \quad a_{ij} < 0, \quad \text{or} \quad j \in P_2{}^n, \quad a_{ij} \geqslant 0. \end{cases}$$

Then $Q(i, P_1{}^n, P_2{}^n) = | \sum_{j=1}^{\infty} g_j{}^n a_{ij} | < (1/n)$. For the fixed i above, let $x = \{a_{ij}\}$ in l_1. Consider the function $f: R \to R$ defined by $f(x) = | x |$. Now define $G: l_\infty \to R$ by $G = f\hat{x}$. Then for $g = \{g_j\}$ in l_∞, $G(g) = | \sum_{j=1}^{\infty} a_{ij} g_j |$. For $i = 1, 2,\ldots$, let e_i be that sequence in l_1 which has 1 in the ith place and 0 elsewhere. Define $F_i : l_\infty \to R$ by $F_i = f\hat{e}_i$ for $i = 1, 2,\ldots$.

Since $\| g^n \| = 1$ for each n, we have $g^n \in S(l_\infty) = S(l_1{}^*)$, which is compact in the weak* topology by Alaoglu's theorem (see [3, p. 424]). Let

$$A = \{g = \{g_j\} \in S(l_\infty): | g_j | = 1 \text{ for all } j\}.$$

Then $g^n \in A$ for all n, so $\inf G(g) = 0$, where the infimum is taken over all $g \in A$. Now $F_i(g) = | g(e_1)|$. Then $\{g \in S(l_\infty): | g(e_i)| = 1\}$ is weak* closed since F_i is weak* continuous. Therefore, $A = \bigcap_i \{g \in S(l_\infty): | g(e_i)| = 1\}$ is weak* compact. Since G is a weak* continuous function on A, there exists $g \in A$ such that $G(g) = 0$, i.e., there exists $g = \{g_j\}$ where $| g_j | = 1$ for all j, and $| \sum_j a_{ij} g_j | = 0$. Let those j which give rise to terms $| a_{ij} |$ for the product $a_{ij} g_j$ be in P_1, and the remaining j in P_2. Then we have exhibited a partition P_1, P_2 such that $Q(i, P_1, P_2) = 0$, a contradiction. Therefore, there does not exist one row i and a sequence of partitions $P_1{}^n$, $P_2{}^n$ such that $Q(i, P_1{}^n, P_2{}^n) < (1/n)$.

It is, therefore, possible to select $i > i_1$ and P_1, P_2 such that $Q(i, P_1, P_2) <$

(1/2). Let i_2 be the smallest $i > i_1$ where this is possible. Assume we have in this way selected $i_1 < \cdots < i_n$ so that for each n, there exists $P_1{}^n$, $P_2{}^n$ such that $Q(i_n, P_1{}^n, P_2{}^n) < (1/n)$. We know there exists i and P_1, P_2 such that $Q(i, P_1, P_2) < [1/(n+1)]$ since the partition property fails to hold for A. Suppose the only possible choice for this i is $i \leqslant i_n$. Then for all $k \geqslant n+1$, there exists $i_k \leqslant i_n$ and $P_1{}^k$, $P_2{}^k$ such that $Q(i_k, P_1{}^k, P_2{}^k) < (1/k)$. For each $s = 1,..., n$, if $i = i_s$, there exists a positive integer k_s such that for all P_1, P_2, $Q(i_s, P_1, P_2) \geqslant (1/k_s)$, since we have previously shown that for any one row i, it is impossible to obtain for all n a partition $P_1{}^n$, $P_2{}^n$ such that $Q(i, P_1{}^n, P_2{}^n) < (1/n)$. Let $k = \max\{k_1, ..., k_n, n+1\}$. Then $k \geqslant n+1$, so there exists $i_k \leqslant i_n$ and $P_1{}^k$, $P_2{}^k$ such that $Q(i_k, P_1{}^k, P_2{}^k) < (1/k)$. However, for all $i \leqslant i_n$ and all P_1, P_2, in particular for i_k and $P_1{}^k$, $P_2{}^k$, we have $Q(i_k, P_1{}^k, P_2{}^k) \geqslant (1/k)$, a contradiction. This completes the inductive argument and proves the lemma.

LEMMA 3.2. *If* $\| B - \lambda A \| = \| B - \lambda_1 A \|$ *for all scalars* λ *in* $[\lambda_1, \lambda_2]$ *(where* $\lambda_1 < \lambda_2$*), then* B *has* λA *as a best approximation in* $[A]$ *for all* λ *in* $[\lambda_1, \lambda_2]$.

Proof. Let $\| B - \lambda A \| = \| B - \lambda_1 A \|$ for all λ in $[\lambda_1, \lambda_2]$. Suppose $\lambda_1 A$ is not a best approximation to B in $[A]$. Then there exists $\mu \in R$ such that $\| B - \mu A \| < \| B - \lambda_1 A \|$. Either $\mu > \lambda_2$ or $\mu < \lambda_1$. Assume $\mu > \lambda_2$ since the other case is similar. Define $\alpha = [(\lambda_2 - \mu)/(\lambda_1 - \mu)]$. Then $\alpha \in (0, 1)$ and $\lambda_2 = \alpha\lambda_1 + (1 - \alpha)\mu$. Hence,

$$\| B - \lambda_2 A \| \leqslant \alpha \| B - \lambda_1 A \| + (1 - \alpha)\| B - \mu A \| < \| B - \lambda_1 A \|,$$

which is a contradiction. This proves the lemma.

THEOREM 3.3. *Let* $A = (a_{ij}) \in [c_0, c_0]$, $A \neq 0$. *Then* $[A]$ *is a Chebychev subspace of* $[c_0, c_0]$ *if and only if* A *satisfies the partition property.*

Proof. Suppose the partition property fails to hold for A. First, let us consider the case where there exists i_0 such that $a_{i_0 j} = 0$ for all j. Define $B = (b_{ij})$, where $b_{i_0 1} = \| A \|$, and $b_{ij} = 0$ for $i \neq i_0$, for all j, and for $i = i_0$, $j = 2, 3, ...$ Then $B \in [c_0, c_0]$ and $\| B \| = \| A \| = \| B - A \|$. For all λ in R, $\sum_{j=1}^{\infty} | b_{i_0 j} - \lambda a_{i_0 j} | = \| A \|$. Hence, for all λ in R, we have $\| B - \lambda A \| \geqslant \| A \| = \| B \|$. Thus, B has 0 and $A \neq 0$ as best approximations in $[A]$, so $[A]$ is not Chebychev.

Now suppose that $\sum_{j=1}^{\infty} | a_{ij} | > 0$ for all i. Since A fails to satisfy the partition property, there are two possible cases by Lemma 3.1.

Case 1. There exists $i_1 < \cdots < i_n < \cdots$ and corresponding partitions $P_1{}^n$, $P_2{}^n$ such that $| \sum_{j \in P_1{}^n} | a_{i_n j} | - \sum_{j \in P_2{}^n} | a_{i_n j} || < (1/n)$. Therefore, for each n there exists $\epsilon^n = \{\epsilon_j^n\}$ in l_∞ with $|\epsilon_j^n| = 1$ for all j and all n such that $0 < \sum_{j=1}^{\infty} \epsilon_j^n a_{i_n j} \leqslant (1/n)$. Then $\lim_{n\to\infty} \sum_{j=1}^{\infty} \epsilon_j^n a_{i_n j} = 0$.

Let $s = \sup \sum_{j=1}^{\infty} | a_{i_n j} |$, the supremum taken over all n, so $0 < s \leqslant \| A \| < \infty$. Now there exists λ_0, $0 < \lambda_0 < 1$ such that $\lambda_0 \| A \| < s$. Define $\sigma_n = s - \sum_{j=1}^{\infty} | a_{i_n j} | \geqslant 0$. Now let $B = (b_{ij})$, where b_{ij} is defined as follows:

$$b_{ij} = \begin{cases} 0 & \text{for } i \neq i_n \quad \text{for any } n = 1, 2,\ldots, \\ \epsilon_j^n \,| a_{i_n j} | + \epsilon_j^n(\sigma_n/i_n) & \text{for } j = 1,\ldots, i_n \quad \text{and} \quad n = 1, 2,\ldots, \\ \epsilon_j^n \,| a_{i_n j} | & \text{for } j > i_n \quad \text{and} \quad n = 1, 2,\ldots. \end{cases}$$

It is easy to see that $\lim_{i\to\infty} b_{ij} = 0$ for all j. The next part of the proof will show that $\| B \| = s < \infty$, which will show by Theorem 2.1 that $B \in [c_0, c_0]$.

Let λ be given, $0 \leqslant \lambda \leqslant \lambda_0$. If $i \neq i_n$ for any n, then

$$\sum_{j=1}^{\infty} | b_{ij} - \lambda a_{ij} | \leqslant \lambda_0 \| A \| < s.$$

If $i = i_n$, we have

$$\begin{aligned} \sum_{j=1}^{\infty} | b_{i_n j} - \lambda a_{i_n j} | &= \sum_{j=1}^{i_n} \big|\, | a_{i_n j} | + (\sigma_n/i_n) - \lambda \epsilon_j^n a_{i_n j} \big| \\ &\quad + \sum_{j=i_n+1}^{\infty} \big|\, | a_{i_n j} | - \lambda \epsilon_j^n a_{i_n j} \big| \\ &= \sum_{j=1}^{i_n} | a_{i_n j} | + \sum_{j=1}^{i_n} (\sigma_n/i_n) - \lambda \sum_{j=1}^{i_n} \epsilon_j^n a_{i_n j} \\ &\quad + \sum_{j=i_n+1}^{\infty} | a_{i_n j} | - \lambda \sum_{j=i_n+1}^{\infty} \epsilon_j^n a_{i_n j} \\ &= s - \lambda \sum_{j=1}^{\infty} \epsilon_j^n a_{i_n j} \leqslant s. \end{aligned}$$

Then since $\lim_{n\to\infty} \sum_{j=1}^{\infty} \epsilon_j^n a_{i_n j} = 0$, we must have $\| B - \lambda A \| = s$ for all λ in $[0, \lambda_0]$. Then by Lemma 3.2, B has λA as a best approximation in $[A]$ for all λ in $[0, \lambda_0]$. Therefore, $[A]$ is not Chebychev.

Case 2. There exists i_0 and partition P_1, P_2 such that

$$\Big| \sum_{j\in P_1} | a_{i_0 j} | - \sum_{j\in P_2} | a_{i_0 j} | \Big| = 0.$$

Therefore, there exists $\{\epsilon_j\}$ in l_∞ with $| \epsilon_j | = 1$ for all j such that $\sum_{j=1}^{\infty} \epsilon_j a_{i_0 j} = 0$.

Let $s = \sum_{j=1}^{\infty} | a_{i_0 j} |$, so $0 < s \leqslant \| A \|$. Then there exists λ_0, $0 < \lambda_0 < 1$ such that $\lambda_0 \| A \| < s$. Let $B = (b_{ij})$ be defined by

$$b_{ij} = \begin{cases} 0 & \text{if } i \neq i_0, \\ \epsilon_j \mid a_{ij} \mid & \text{if } i = i_0. \end{cases}$$

Then $B \in [c_0, c_0]$. Let $\lambda \in [0, \lambda_0]$. For $i \neq i_0$, $\sum_{j=1}^{\infty} | b_{ij} - \lambda a_{ij} | < s$. If $i = i_0$,

$$\sum_{j=1}^{\infty} | b_{i_0 j} - \lambda a_{i_0 j} | = \sum_{j=1}^{\infty} \| a_{i_0 j} | - \lambda \epsilon_j a_{i_0 j} | = s.$$

Therefore, $\| B - \lambda A \| = s$ for all λ in $[0, \lambda_0]$. By Lemma 3.2, $[A]$ is not Chebychev.

We must now show sufficiency. Without loss of generality, assume $\| A \| = 1$. Suppose $[A]$ is not Chebychev. It is easy to see that there exist B in $[c_0, c_0]$, $\| B \| = 1$, and $\lambda > 0$ such that B has 0 and $\pm\lambda A \neq 0$ as best approximations in $[A]$. Let $\epsilon > 0$ and $\epsilon' = \epsilon\lambda$. Then by Theorem 2.3 there exist m positive integers $k_1, \ldots, k_m$, m l_∞ sequences $f^1, \ldots, f^m$ with $\| f^i \| = 1$, $i = 1, \ldots, m$, and m scalars $r_1, \ldots, r_m$ with $r_i > 0$, $i = 1, \ldots, m$ and $\sum_{i=1}^{m} r_i = 1$ such that

(i) $\sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j^i a_{k_i j} = 0$,

(ii) $| \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j^i b_{k_i j} - 1 | < \epsilon'$.

Then there exists i such that

$$\left| \sum_{j=1}^{\infty} f_j^i b_{k_i j} - 1 \right| < \epsilon'. \tag{1}$$

Define $\{g_j\}$ in l_∞ by

$$g_j = \begin{cases} b_{k_i j} / | b_{k_i j} | & \text{if } b_{k_i j} \neq 0, \\ 1 & \text{if } b_{k_i j} = 0. \end{cases}$$

Then $g_j b_{k_i j} = | b_{k_i j} |$ and $| g_j | = 1$ for all j since the scalars are real. Since $\| f^i \| = 1$, it follows that

$$\sum_{j=1}^{\infty} g_j b_{k_i j} \geqslant \sum_{j=1}^{\infty} f_j^i b_{k_i j}. \tag{2}$$

Next we will show that $| \sum_{j=1}^{\infty} g_j a_{k_i j} | < (\epsilon'/\lambda)$ for the selected i. If $\sum_{j=1}^{\infty} g_j a_{k_i j} = 0$, there is nothing to prove. Suppose $\lambda \sum_{j=1}^{\infty} g_j a_{k_i j} > 0$. Now $| \sum_{j=1}^{\infty} g_j (b_{k_i j} \pm \lambda a_{k_i j}) | \leqslant \| B \pm \lambda A \| = \| B \| = 1$ since 0 and $\pm\lambda A$ are best approximations to B in $[A]$. Then using (1) and (2), we have

$1 \geqslant \sum_{j=1}^{\infty} g_j(b_{k_i j} + \lambda a_{k_i j}) > 1 - \epsilon' + \lambda \sum_{j=1}^{\infty} g_j a_{k_i j}$. Therefore,

$$\left| \sum_{j=1}^{\infty} g_j a_{k_i j} \right| < (\epsilon'/\lambda) = \epsilon.$$

If $-\lambda \sum_{j=1}^{\infty} g_j a_{k_i j} > 0$, the result follows in a similar fashion. Let P_1 be the set of those j where $g_j a_{k_i j} = | a_{k_i j} |$, and let P_2 consist of the remaining j. Then we have exhibited k_i and a partition P_1, P_2 such that $\left| \sum_{j \in P_1} | a_{k_i j} | - \sum_{j \in P_2} | a_{k_i j} | \right| < \epsilon$. Hence, A fails to satisfy the partition property, and the theorem is proved.

4. Results for Finite Dimensional Subspaces

We return in this section to an arbitrary finite dimensional subspace M of $[c_0, c_0]$, and now present a necessary condition for M to be non-Chebychev.

THEOREM 4.1. *Let $M = [A_1, ..., A_n]$ be a non-Chebychev subspace of $[c_0, c_0]$, where $A_k = (a_{ij}^k) \in [c_0, c_0]$ with $\| A_k \| = 1$, $k = 1,..., n$. Then there exists $A = (a_{ij})$ in M, $\| A \| = 1$, such that given $\epsilon > 0$, there exist m positive integers $k_1, ..., k_m$ and $\alpha = (\alpha_{ij})$ in $M^{\perp}$ with $\alpha_{ij} = 0$ for $i \neq k_1, ..., k_m$, $\sup_j | \alpha_{ij} | < \infty$ for all i and $\sum_{i=1}^{m} \sup_j | \alpha_{k_i j} | = 1$ such that*

(i) *if $\beta \in [c_0, c_0]^*$ and $\| \alpha \pm \beta \| \leqslant 1$, then $| \beta(A)| < \epsilon$,*

(ii) $\sum_{i=1}^{\infty} | \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} | < \epsilon$.

Proof. Since M is non-Chebychev, it follows that there exists B in $[c_0, c_0]$ such that B has 0 and $\pm A \neq 0$ as best approximations in M, with $\| A \| = 1$. Thus, $\| B \| = \| B - A \| = \| B + A \|$. This is the required A. Let $\epsilon > 0$. Then by Theorem 2.3 there exist m positive integers $k_1, ..., k_m$, m l_∞ sequences $f^1, ..., f^m$ with $\| f^i \| = 1$, $i = 1,..., m$ and m scalars $r_1, ..., r_m$ with $r_i > 0$, $i = 1,..., m$ and $\sum_{i=1}^{m} r_i = 1$ such that

(i') $\sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j^i a_{k_i j}^k = 0 \qquad k = 1,..., n,$

(ii') $| \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j^i b_{k_i j} - \| B \|| < (\epsilon/2).$

Define $\alpha = (\alpha_{ij})$ by $\alpha_{k_i j} = r_i f_j^i$ for $i = 1,..., m$ and $\alpha_{ij} = 0$ for $i \neq k_1, ..., k_m$. Then $\sup_j | \alpha_{ij} | < \infty$ for all i and $\sum_{i=1}^{m} \sup_j | \alpha_{k_i j} | = 1$. For $T = (t_{ij}) \in [c_0, c_0]$, let $\alpha(T) = \sum_{i=1}^{m} r_i \sum_{j=1}^{\infty} f_j^i t_{k_i j}$. Then it is easy to see that $\alpha \in [c_0, c_0]^*$. By (i'), $\alpha \in M^{\perp}$. By (ii') we have

$$| \alpha(B) - \| B \|| < (\epsilon/2). \tag{1}$$

To prove (i), let $\beta \in [c_0, c_0]^*$ with $\| \alpha \pm \beta \| \leqslant 1$. Then

$$(\alpha \pm \beta)(B) \leqslant \| B \| \| \alpha \pm \beta \| \leqslant \| B \|,$$

so by (1), $| \beta(B)| < (\epsilon/2)$. Similarly, since $\alpha \in M^{\perp}$, we obtain

$$\alpha(B) \pm \beta(B - A) \leqslant \| B - A \| = \| B \|.$$

Thus, $| \beta(B - A)| < (\epsilon/2)$. It follows that $|\beta(A)| < \epsilon$.

We must now show (ii). Let

$$P = \left\{ i: \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} \geqslant 0 \right\},\ P' = \left\{ i: \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} > 0 \right\},\ \text{and}\ N = \left\{ i: \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} < 0 \right\}.$$

Since $\alpha \in M^{\perp}$, $\alpha(A) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} = 0$. Thus, if $N = \phi$, then $P' = \phi$, and conversely. In this case (ii) holds trivially. Therefore, assume $P' \neq \phi$ and $N \neq \phi$. Now $\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} + \sum_{i \in N} \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} = 0$. Therefore, $\sum_{i \in P} | \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} | = \sum_{i \in N} | \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} |$. Now suppose (ii) is false. Then

$$\sum_{i \in P} \left| \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} \right| \geqslant (\epsilon/2). \tag{2}$$

Also, $\sum_{i \in N} | \sum_{j=1}^{\infty} \alpha_{ij} a_{ij} | \geqslant (\epsilon/2)$. For each $i = 1, 2, \ldots$, let $\lambda_i = \sup | \alpha_{ij} | \geqslant 0$, the supremum taken over all j. Let $\lambda_P = \sum_{i \in P} \lambda_i > 0$ and $\lambda_N = \sum_{i \in N} \lambda_i > 0$. Then $\lambda_P + \lambda_N = 1$. Let $E_1 = \sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij} b_{ij}$ and $E_2 = \sum_{i \in N} \sum_{j=1}^{\infty} \alpha_{ij} b_{ij}$. Then $E_1 + E_2 = \alpha(B) > \| B \| - (\epsilon/2)$ by (1). This implies that either (a) $E_1 > \lambda_P[\| B \| - (\epsilon/2)]$ or (b) $E_2 > \lambda_N[\| B \| - (\epsilon/2)]$ must hold. Suppose (a) is true. Then using (2), we obtain

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}(b_{ij} + a_{ij}) > \lambda_P \| B \| - (\epsilon/2)\, \lambda_P + (\epsilon/2) > \lambda_P \| B \| = \lambda_P \|B + A \|.$$

But for each i in P, $\sum_{j=1}^{\infty} \alpha_{ij}(b_{ij} + a_{ij}) \leqslant \lambda_i \| B + A \|$. Thus,

$$\sum_{i \in P} \sum_{j=1}^{\infty} \alpha_{ij}(b_{ij} + a_{ij}) \leqslant \lambda_P \| B + A \|.$$

Thus, we have been led to a contradiction. If (b) holds, we obtain a contradiction in a similar manner. Therefore, (ii) is proved.

THEOREM 4.2. *Let $M = [A_1, \ldots, A_n]$ be an n-dimensional Chebychev subspace of $X = [c_0, c_0]$. Let $i_1 < \cdots < i_s$ denote a fixed but arbitrary finite* *number of rows. For* $T = (t_{ij})$ *in* X, *define* $\bar{T} = (t_{i_kj}) \in (l_1 \times \cdots \times l_1)_\infty$ (s summands). Let $\bar{X} = \{\bar{T}: T \in X\}$. Then $\bar{M} = [\bar{A}_1, ..., \bar{A}_n]$ is an n-dimensional Chebychev subspace of $\bar{X}$.

Proof. First, we will show that $\bar{M}$ is a Chebychev subspace of $\bar{X}$. Suppose not. Then there exists $\bar{B} \in \bar{X}$, $\|\bar{B}\| = 1$ such that $\bar{B}$ has 0 and $\bar{A}' \neq 0$ as best approximations in $\bar{M}$. Hence, there exist scalars $\lambda_1, ..., \lambda_n$ not all 0 such that $\bar{A}' = \sum_{i=1}^n \lambda_i \bar{A}_i$. Let $A' = \sum_{i=1}^n \lambda_i A_i \neq 0$. Let $A = \lambda A' \neq 0$, where $\lambda = 1$ if $\| A' \| \leqslant 1$ and $\lambda = (1/\| A' \|)$ if $\| A' \| > 1$. Since $\bar{M}$ is convex, the set of best approximations to $\bar{B}$ in $\bar{M}$ is convex. Hence, $\bar{A}$ is a best approximation to $\bar{B}$ in $\bar{M}$, and $\| \bar{B} - \bar{A} \| = \| \bar{B} \| = 1$. Define $B = (b_{ij})$ in X by

$$B = \begin{cases} \bar{B} & \text{on rows } i_1, ..., i_s, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\| B \| = \| \bar{B} \| = 1$. Since $\| A \| \leqslant 1$, we have $\| B - A \| = 1 = \| B \|$.

Now since $\bar{B}$ has 0 as a best approximation in $\bar{M}$, there exists $\bar{f} \in \bar{M}^\perp$, $\|\bar{f}\| = 1$, and $\bar{f}(\bar{B}) = \| \bar{B} \|$ by Theorem 1.2. Define f on X by $f(T) = \bar{f}(\bar{T})$ for all T in X. It is easy to see that $f \in M^\perp$, $f(B) = \| B \|$, and $\|f\| = 1$. Then by Theorem 1.2, B has 0 as a best approximation in M. Since $\| B \| = \| B - A \|$, B has 0 and $A \neq 0$ as best approximations in M. But this is a contradiction, since M is Chebychev. Therefore, $\bar{M}$ is a Chebychev subspace of $\bar{X}$.

Now suppose $\bar{M}$ is not n-dimensional. Then there exists A in M, $\| A \| = 1$, with $\bar{A} = 0$. Since $\dim \bar{M} < \infty = \dim \bar{X}$, there exists $\bar{B}$, $\| \bar{B} \| = 1$, such that $\bar{B}$ has 0 (and, hence, $\bar{A}$) as a best approximation in $\bar{M}$. Define $B = (b_{ij})$ in X as in the first part of this proof. By duplicating the steps following that definition of B, we can show that B has 0 and $A \neq 0$ as best approximations in M. Again we obtain a contradiction, thus showing that $\bar{M}$ is n-dimensional and completing the proof of this theorem.

Theorem 4.2 can be utilized to obtain a sufficient condition for a finite dimensional subspace of $[c_0, c_0]$ to be non-Chebychev. If the spanning matrices are dependent on at least one row, then the subspace is not Chebychev. This is stated in the following corollary.

COROLLARY 4.3. *Let* $M = [A_1, ..., A_n]$ *be an n-dimensional subspace of* $[c_0, c_0]$, *where* $A_k = (a_{ij}^k)$, $k = 1, ..., n$. *Suppose there exists a row* i_0 *and scalars* $\lambda_1, ..., \lambda_n$ *not all* 0 *such that* $\lambda_1 a_{i_0j}^1 + \cdots + \lambda_n a_{i_0j}^n = 0$ *for all* j. *Then* M *is not Chebychev.*

Proof. Let $s = 1$ in Theorem 4.2, so we have one row i_1. Then since $\bar{M}$ is not n-dimensional, M is not Chebychev.

The *adjoint* T^* of a bounded linear operator T from c_0 to c_0 is the mapping from c_0^* to c_0^* defined by $T^*y^* = y^*T$. By [5, p. 201] c_0^* can be identified with l_1. By [3, p. 478] we can see that $T^* \in [l_1, l_1]$ and $\| T^* \| = \| T \|$. Recall that l_1^* can be identified with l_∞. Hence, the second adjoint $T^{**} \in [l_\infty, l_\infty]$. Now $\hat{c}_0 = c_0 \subseteq l_\infty$. It is known by [3, p. 479] that $T^{**}: l_\infty \to l_\infty$ is an extension of T, i.e., for $x \in c_0$, $T^{**}(x) = T(x)$.

Let T be represented by the infinite matrix of scalars (α_{ij}), so that by Theorem 2.1, if $x = \{x_i\} \in c_0$, $y = \{y_i\} \in c_0$, then $Tx = y$ can be expressed by the equations

$$y_i = \sum_{j=1}^{\infty} \alpha_{ij} x_j \qquad i = 1, 2, \ldots.$$

The norm of T is given by $\| T \| = \sup_i \sum_{j=1}^{\infty} | \alpha_{ij} |$. Then by [5, p. 220], the matrix (α_{ij}) also defines a bounded linear operator T' on l_∞ into l_∞ with the same defining equations and same norm. Thus, T' is also an extension of T, i.e., for $x \in c_0$, $T'(x) = T(x)$.

THEOREM 4.5. *Let the bounded linear operator T on c_0 into c_0 be represented by the infinite matrix of scalars (α_{ij}), and let T' represent (α_{ij}) considered as a bounded linear operator from l_∞ to l_∞. Then $T' = T^{**}$.*

Proof. Let $b = \{b_j\} \in l_\infty$ and let $T'(b) = z = \{z_i\} \in l_\infty$, so $z_i = \sum_{j=1}^{\infty} \alpha_{ij} b_j$ for $i = 1, 2, \ldots$. Let $y^* = \{y_i^*\} \in l_1$. For each $j = 1, 2, \ldots$, define e_j to be that sequence in c_0 which has 1 in the jth place and 0 elsewhere. Then $T(e_j) = (\alpha_{1j}, \alpha_{2j}, \ldots)$. Therefore, $T^*y^*(e_j) = \sum_{i=1}^{\infty} y_i^* \alpha_{ij}$. Let $x = \{x_j\} \in c_0$. Then $x = \sum_{j=1}^{\infty} x_j e_j$. Hence,

$$T^*y^*(x) = \sum_{j=1}^{\infty} x_j (T^*y^*)(e_j) = \sum_{j=1}^{\infty} x_j \sum_{i=1}^{\infty} y_i^* \alpha_{ij}. \tag{1}$$

Now consider $f = \{f_j\}$ where $f_j = \sum_{i=1}^{\infty} y_i^* \alpha_{ij}$ for $j = 1, 2, \ldots$. Then we have $\sum_{j=1}^{\infty} | f_j | < \infty$, where the interchange of limits is justified by a standard theorem (see, e.g., [1, p. 398]). Hence, $f \in l_1$. We also know $T^*y^* \in l_1$. Moreover, for any x in c_0, $f(x) = (T^*y^*)(x)$ by (1). Thus, $T^*y^* = f$. Now, justifying the interchange of limits in the same manner as above, we obtain

$$b(T^*y^*) = \sum_{j=1}^{\infty} b_j \sum_{i=1}^{\infty} y_i^* \alpha_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} b_j y_i^* \alpha_{ij} = \sum_{i=1}^{\infty} y_i^* z_i.$$

Therefore, $(T^{**}b)(y^*) = b(T^*y^*) = z(y^*) = (T'b)(y^*)$. Since this holds for all y^* in l_1, we must have $T^{**}b = T'b$. But b was arbitrary in l_∞. Therefore, $T' = T^{**}$, and the proof is completed.

We conclude by noting that Theorem 4.5 permits the expression of the principal results in this paper in terms of the second adjoint of a bounded linear operator, rather than in terms of the operator's matrix representation.

As an example of this, we give an equivalent formulation of Theorem 3.3 in the following corollary.

COROLLARY 4.6. *Let $A \in [c_0, c_0]$, $A \neq 0$. Then $[A]$ is a Chebychev subspace of $[c_0, c_0]$ if and only if there exists $\delta > 0$ such that if $x = \{x_j\} \in l_\infty$ with $|x_j| = 1$ for all j, and $A^{**}(x) = \{y_i\}$, then $|y_i| \geqslant \delta$ for all i.*

Proof. By Theorem 4.5, if $A = (a_{ij})$, *then* $A^{**} = (a_{ij})$, so $y_i = \sum_{j=1}^{\infty} a_{ij} x_j$. Then the given condition holds if and only if A satisfies the partition property. The result then follows by Theorem 3.3.

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